http://dx.doi.org/10.18623/rvd.v23.5536

A FUZZY NEOTERIC INTEGRAL TRANSFORMATION

UMA TRANSFORMAÇÃO INTEGRAL NEOTÉRICA DIFUSA

Article received on: 12/11/2025
Article accepted on: 3/12/2026

Zainab R Zainy*
*University of Kufa (OuK), Najaf, Iraq
zainabr.zainy@student.uokufa.edu.iq

Athraa N Albukhuttar*
*University of Kufa (OuK), Najaf, Iraq
athraan.kadhim@uokufa.edu.iq

The authors declare that there is no conflict of interest

Abstract
In this work, we examine the Neoteric Integral Transformation within a fuzzy framework, hereafter denoted as the fuzzy neoteric transformation (FNT). Additionally, we present several findings concerning the characteristics of the fuzzy neoteric transformation, including linearity, preservation of fuzzy derivatives, and the convolution property. To demonstrate the efficiency of the (FNT), a comprehensive method for solving higher- order fuzzy differential equation (FDEs) is developed. Moreover, a numerical example is introduced to highlight the practical application of the (FNT).

Keywords: Fuzzy Numbers. Fuzzy Integral Transform. Fuzzy Differential Equation. Neoteric Transform.

Resumo
Neste trabalho, examinamos a Transformação Integral Neotérica dentro de uma estrutura difusa, doravante denominada transformação neotérica difusa (FNT). Além disso, apresentamos várias descobertas relativas às características da transformação neotérica difusa, incluindo linearidade, preservação de derivadas difusas e a propriedade de convolução. Para demonstrar a eficiência da (FNT), é desenvolvido um método abrangente para resolver equações diferenciais difusas (FDEs) de ordem superior. Além disso, é apresentado um exemplo numérico para destacar a aplicação prática da (FNT).

Palavras-chave: *Números Difusos. Transformada Integral Difusa. Equação Diferencial Difusa. Transformada Neotérica.*

1 INTRODUCTION

Integral transforms are powerful mathematical tools that play a significant role in solving differential, integral, and integral-differential equations. Classical transforms such as the Laplace transform and Fourier transform have been extensively applied in science and engineering [1-3]. In recent years, new integral transforms have been proposed to generalize and broaden the application of existing methods across various scientific fields [4-8].

On the other hand, fuzzy set theory and fuzzy numbers provide a mathematical framework to model uncertainty and imprecision, which are inherent in many real-world systems. By combining integral transforms with fuzzy analysis, one can develop new techniques for solving fuzzy differential equations and uncertain models [9-12]. In this paper, we introduce a fuzzy version (FNT) of a recently defined integral transform [13].

2 DEFINITIONS AND CORE NOTIONS

Definition1: [12] (parametric form of fuzzy number): let a fuzzy number m in parametric form be represented as a pair ($\underline{m}(\varepsilon), \overline{m}(\varepsilon)$) of functions which satisfy the prerequisites:

1) $\underline{m}(\varepsilon)$ is a bounded non decreasing left continuous function in (0,1] and right continuous at 0.
2) $\overline{m}(\varepsilon)$ is a bounded non-increasing left continuous function in (0,1] and right continuous at 0.
3) $\underline{m}(\varepsilon) \leq \overline{m}(\varepsilon) \qquad 0 \leq \varepsilon \leq 1.$

Definition 2: [12] For arbitrary $\delta = \left(\underline{\delta}(\varepsilon), \overline{\delta}(\varepsilon)\right)$, $\mu = (\underline{\mu}(\varepsilon), \overline{\mu}(\varepsilon) \quad \forall\ \varepsilon \in$ [0,1] we explicate

1) Addition $\qquad \delta \oplus \mu = \left(\underline{\delta}(\varepsilon) + \underline{\mu}(\varepsilon), \overline{\delta}(\varepsilon) + \overline{\mu}(\varepsilon)\right).$
2) Subtraction $\qquad \delta \ominus \mu = \left(\underline{\delta}(\varepsilon) - \overline{\mu}(\varepsilon), \overline{\delta}(\varepsilon) - \underline{\mu}(\varepsilon)\right).$
3) Scalar multiplication $\qquad b \odot \mu = \begin{cases} \left(b\underline{\mu}(\varepsilon), b\overline{\mu}(\varepsilon)\right) & b \geq 0 \\ \left(b\overline{\mu}(\varepsilon), b\underline{\mu}(\varepsilon)\right) & b < 0 \end{cases}$

Definition 3.[12] For arbitrary $\delta = \left(\underline{\delta}(\varepsilon), \overline{\delta}(\varepsilon)\right)$, $\mu = (\underline{\mu}(\varepsilon), \overline{\mu}(\varepsilon) \ \ \forall\ \varepsilon \in [0,1]$ we explicate $\ominus\, \delta \ and - (\ominus\, \delta)$, in the following manner :

- $\ominus\, \delta = (-\underline{\delta}(\varepsilon), -\overline{\delta}(\varepsilon))$
- $-(\ominus\, \delta) = (\overline{\delta}(\varepsilon), \underline{\delta}(\varepsilon))$
- $-(\ominus\ominus\, \delta) = (-\overline{\delta}(\varepsilon), -\underline{\delta}(\varepsilon))$
- $\delta \ominus \mu = (\underline{\delta}(\varepsilon) - \underline{\mu}(\varepsilon), \overline{\delta}(\varepsilon) - \overline{\delta}(\varepsilon)$

Definition 4.[9] let $\mathfrak{U}(y)$ be fuzzy – valued mapping defined over $[c,\infty)$, expressed in the form $(\underline{\mathfrak{U}}(y,₹),\overline{\mathfrak{U}}(y,₹))$. For any chosen $₹ \in [0,1]$, suppose that $\underline{\mathfrak{U}}(y,₹)$ *and* $\overline{\mathfrak{U}}(y,₹)$ are Riemann-integral in $[c,d]$. Assume the existence of two positive constants $\underline{\mathbb{K}}(₹)$ *and* $\overline{\mathbb{K}}(₹)$ such that

$$\int_c^d |\underline{\mathfrak{U}}(y,₹)|\,dy \le \underline{\mathbb{K}}(₹),\ \int_c^d |\overline{\mathfrak{U}}(y,₹)|\,dy \le \overline{\mathbb{K}}(₹) \quad \forall d \ge c$$

Hence, $\mathfrak{U}(y)$ is fuzzy improper Riemann integrable over the interval $[c,\infty]$, and the resulting improper fuzzy Riemann-integral constitutes a fuzzy number . In addition, we obtain**:**

$$\int_c^\infty \mathfrak{U}(y)\,dy = \left(\int_c^\infty \underline{\mathfrak{U}}(y,₹)\,dy, \int_c^\infty \overline{\mathfrak{U}}(y,₹)\,dy\right).$$

Remark 1: We denoted by $\mathbb{R}_{\mathfrak{f}}$ the collection of all fuzzy numbers on the set of real numbers $\mathbb{R}$.

Theorem 1.[14] assume $\mathfrak{U}:(c,d) \to \mathbb{R}_{\mathfrak{f}}$ and let $y \in (c,d)$. We assert that $\mathfrak{U}$ **is strongly generalized differentiable** at y if $\exists\ \mathfrak{U}'(y) \in \mathbb{R}_{\mathfrak{f}}$ s.t. one of the following holds,

(a) $\forall\, ƀ > 0$, where ($ƀ$ is small constant), the limits are exist:

$$\lim_{ƀ\to 0}\frac{\mathfrak{U}(y+ƀ)\ominus\mathfrak{U}(y)}{ƀ} = \lim_{ƀ\to 0}\frac{\mathfrak{U}(y)\ominus\mathfrak{U}(y-ƀ)}{ƀ} = \mathfrak{U}'(y)$$

(b) $\forall\, ƀ > 0$, where ($ƀ$ is small constant), the limits are exist:

$$\lim_{ƀ\to 0}\frac{\mathfrak{U}(y)\ominus\mathfrak{U}(y+ƀ)}{-ƀ} = \lim_{ƀ\to 0}\frac{\mathfrak{U}(y-ƀ)\ominus\mathfrak{U}(y)}{-ƀ} = \mathfrak{U}'(y)$$

(c) $\forall\, ƀ > 0$, where ($ƀ$ is small constant), the limits are exist:

$$\lim_{ƀ\to 0}\frac{\mathfrak{U}(y+ƀ)\ominus\mathfrak{U}(y)}{ƀ} = \lim_{ƀ\to 0}\frac{\mathfrak{U}(y-ƀ)\ominus\mathfrak{U}(y)}{-ƀ} = \mathfrak{U}'(y)$$

$(d)\ \forall\, \text{ʉ} > 0$, where ($\mathcal{u}$ is small constant), the limits are exist:

$$\lim_{\text{ʉ}\to 0}\frac{\mathfrak{U}(y) \ominus \mathfrak{U}(y+\text{ʉ})}{-\text{ʉ}} = \lim_{\text{ʉ}\to 0}\frac{\mathfrak{U}(y) \ominus \mathfrak{U}(y-\text{ʉ})}{\text{ʉ}} = \mathfrak{U}'(y)$$

Remark 2:

In this work, we restrict our attention to cases $(a)\ and\ (b)$ of the strongly generalized differentiability introduced by Bede and Gal [14]. Later, Chalco-Cano and Roman-Flores [15] indicated that Cases $(a)\ and\ (b)$ are the most relevant, since cases $(c)\ and\ (d)$ only arise at isolated points.

Theorem 2.[10] Suppose $\mathfrak{U}: \mathbb{R} \to \mathbb{R}_{\mathfrak{f}}$ is continuous fuzzy – valued function, and denote

$$\mathfrak{U}(y) = \left(\underline{\mathfrak{U}}(y,\xi), \overline{\mathfrak{U}}(y,\xi)\right)\ \forall\, \xi \in [0,1].$$

- If $\mathfrak{U}$ is (a) – differentiable, the functions $\underline{\mathfrak{U}}(y,\xi)\ and\ \overline{\mathfrak{U}}(y,\xi)$ are differentiable , and their derivatives coincide with the lower and upper bounds of $\mathfrak{U}'(y)$, i.e.,

$$\mathfrak{U}'(y) = \left(\underline{\mathfrak{U}'}(y,\xi), \overline{\mathfrak{U}}'(y,\xi)\right).$$

- If $\mathfrak{U}$ is (b) – differentiable, the functions $\underline{\mathfrak{U}}(y,\xi)\ and\ \overline{\mathfrak{U}}(y,\xi)$ are also differentiable, and one has

$$\mathfrak{U}'(y) = \left(\overline{\mathfrak{U}}'(y,\xi), \underline{\mathfrak{U}'}(y,\xi)\right).$$

Theorem 3.[10] Assume $\mathfrak{U}: (c,d) \to \mathbb{R}_{\mathfrak{f}}$ and let $y \in (c,d)$. We assert that $\mathfrak{U}$ **is strongly generalized differentiable** at y if $\exists\, \mathfrak{U}''(y) \in \mathbb{R}_{\mathfrak{f}}$ s.t. one of the following holds

$(a)\ \forall \text{ʉ} > 0$, where (ʉ is small constant), the limits are exist:

$$\lim_{\text{ʉ}\to 0}\frac{\mathfrak{U}'(y+\text{ʉ})\ominus\mathfrak{U}'(y)}{\text{ʉ}}=\lim_{\text{ʉ}\to 0}\frac{\mathfrak{U}'(y)\ominus\mathfrak{U}'(y-\text{ʉ})}{\text{ʉ}}=\mathfrak{U}''(y)$$

(b) $\forall$ ʉ > 0 , where (ʉ is small constant), the limits are exist:

$$\lim_{\text{ʉ}\to 0}\frac{\mathfrak{U}'(y)\ominus\mathfrak{U}'(y+\text{ʉ})}{-\text{ʉ}}=\lim_{\text{ʉ}\to 0}\frac{\mathfrak{U}'(y-\text{ʉ})\ominus\mathfrak{U}'(y)}{-\text{ʉ}}=\mathfrak{U}''(y)$$

(c) $\forall$ ʉ > 0 , where (ʉ is small constant), the limits are exist:

$$\lim_{\text{ʉ}\to 0}\frac{\mathfrak{U}'(y+\text{ʉ})\ominus\mathfrak{U}'(y)}{\text{ʉ}}=\lim_{\text{ʉ}\to 0}\frac{\mathfrak{U}'(y-\text{ʉ})\ominus\mathfrak{U}'(y)}{-\text{ʉ}}=\mathfrak{U}''(y)$$

(d) $\forall$ ʉ > 0 , where (ʉ is small constant), the limits are exist:

$$\lim_{\text{ʉ}\to 0}\frac{\mathfrak{U}'(y)\ominus\mathfrak{U}'(y+\text{ʉ})}{-\text{ʉ}}=\lim_{\text{ʉ}\to 0}\frac{\mathfrak{U}'(y)\ominus\mathfrak{U}'(y-\text{ʉ})}{\text{ʉ}}=\mathfrak{U}''(y)$$

Theorem 4.[12] Suppose the differentiable fanctions $\mathfrak{U}(y)$ and $\mathfrak{U}'(y)$ are both fuzzy that

have $\mathfrak{U}(y)=\left(\underline{\mathfrak{U}}(y,\xi),\overline{\mathfrak{U}}(y,\xi)\right)\ \forall\xi\in[0,1]$:

1) If both $\mathfrak{U}(y)$, $\mathfrak{U}'(y)$ are ((a) − differentiable) , or if both $\mathfrak{U}(y)$, $\mathfrak{U}'(y)$ are ((b) − differentiable) , then the lower and upper functions $\underline{\mathfrak{U}}(y,\xi),\overline{\mathfrak{U}}(y,\xi)$ possess first-order , second-order derivatives and the second-order derivatives of $\mathfrak{U}(y)$ is expressed as :

   $\mathfrak{U}''(y)=(\underline{\mathfrak{U}}''(y,\xi),\overline{\mathfrak{U}}''(y,\xi))$

2) If $\mathfrak{U}(y)$ $is\ (a)-differentiable\ and$ $\mathfrak{U}'(y)$ is $(b)-differentiable$, or if $\mathfrak{U}(y)$ is $(b)-differentiableand$ $\mathfrak{U}'(y)$ is $(a)-differentiable$, then the functions $\underline{\mathfrak{U}}(y,\xi),\overline{\mathfrak{U}}(y,\xi)$ have well-defined first-order and second-order derivatives, and the second derivative of $\mathfrak{U}(y)$ is expressed as :

$$\mathfrak{U}''(y) = (\,\overline{\mathfrak{U}}''(y,\xi), \underline{\mathfrak{U}}''(y,\xi)\,)$$

3 DEFINITION AND INTRINSIC ATTRIBUTES OF FUZZY NEOTERIC INTEGRAL TRANSFORM

To derive the required results, certain definitions are necessary. In this work, $\widetilde{\mathbb{T}}_{f}\{\mathfrak{U}(y)\}$ and $\widetilde{\mathbb{F}}_{f}(\mathfrak{s},\mathfrak{r})$ shall serve as the symbols representing the fuzzy neoteric general integral transformation.

The fuzzy Neoteric Integral transform is regarded as an advanced mathematical tool designed to simplify fuzzy ordinary differential equations (FODE) by converting them into algebraic forms that are more suitable for analysis. This technique replaces differential operations with their equivalent integral and computational operations, thereby providing a practical framework for addressing mathematical problems of a fuzzy nature. Moreover, the transform enables systematic treatment of first order derivatives through precise transform formulas, making it one of the fundamental operational methods in both applied and theoretical research within modern mathematics.

Definition 5. Consider $\mathfrak{U}: \mathbb{R} \rightarrow \mathbb{R}_{\mathfrak{f}}$ as a continuous fuzzy – valued function . Provided that $\mathfrak{U}(y) \odot e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}y}$ is improperly fuzzy Riemann integral over $[0,\infty)$, it is designated as the fuzzy neoteric integral transformation and represented by

$$\widetilde{\mathbb{T}}_{f}\{\mathfrak{U}(y)\} = \int_0^\infty \mathfrak{U}(y) \odot e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}y}\, dy = \widetilde{\mathbb{F}}(\mathfrak{s},\mathfrak{r})$$

Where $y \geq 0$, $\mathfrak{y}(\mathfrak{s}), \mathbb{v}(\mathfrak{r})$are positive real functions and $\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \neq 0.$

By virtue of Theorem 1

$$\int_0^\infty \mathfrak{U}(y) \odot e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}y}\, dy = \left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\int_0^\infty \underline{\mathfrak{U}}(y,\xi) e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}y}\, dy\ ,\ \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\int_0^\infty \overline{\mathfrak{U}}(y,\xi)\, dy\ \right)$$

From the framework of the classical neoteric transformation , it results that

$$\mathbb{T}\{\underline{\mathfrak{U}}(y,\xi)\} = \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\int_0^\infty \underline{\mathfrak{U}}(y,\xi)e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}y}\,dy$$

Likewise:

$$\mathbb{T}\{\overline{\mathfrak{U}}(y,\xi)\} = \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\int_0^\infty \overline{\mathfrak{U}}(y,\xi)e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}y}\,dy$$

Eventually, we derive:

$$\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}(y)\} = (\mathbb{T}\{\underline{\mathfrak{U}}(y,\xi)\}, \mathbb{T}\{\overline{\mathfrak{U}}(y,\xi)\})$$

3.1 Linearity and convolution of fuzzy neoteric integral transformation

Let $\mathfrak{U}: \mathbb{R} \rightarrow \mathbb{R}_{\mathfrak{f}}$, $\mathfrak{j}: \mathbb{R} \rightarrow \mathbb{R}_{\mathfrak{f}}$ as a continuous fuzzy – valued function, $\alpha_1 and\ \alpha_2$ are any chosen constants, then :-

1. $\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\alpha_1\mathfrak{U}(y)\} = \alpha_1 \odot \widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}(y)\}$.
2. $\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\alpha_1\mathfrak{U}(y) \oplus \alpha_2\mathfrak{j}(y)\} = \alpha_1 \odot \widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}(y)\} \oplus \alpha_2 \odot \widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{j}(y)\}$.

proof 1. $\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\alpha_1\mathfrak{U}(y)\} = (\mathbb{T}\{\alpha_1\underline{\mathfrak{U}}(y,\xi)\}, \mathbb{T}\{\alpha_1\overline{\mathfrak{U}}(y,\xi)\})$ where $\xi \in [0,1]$

$$= \left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\int_0^\infty (\alpha_1\,\underline{\mathfrak{U}}(y,\xi))e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}y}\,dy, \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\int_0^\infty (\alpha_1\,\overline{\mathfrak{U}}(y,\xi))e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}y}\,dy\right)$$

$$= \alpha_1(\mathbb{T}\{\underline{\mathfrak{U}}(y,\xi)\}, \mathbb{T}\{\overline{\mathfrak{U}}(y,\xi)\}) = \alpha_1 \odot \widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}(y)\} \quad \square$$

Proof 2. let $\mathfrak{U}(y) = \left(\underline{\mathfrak{U}}(y,\xi), \overline{\mathfrak{U}}(y,\xi)\right)$ $\quad and \quad$ $\mathfrak{j}(y) = \left(\underline{\mathfrak{j}}(y,\xi), \overline{\mathfrak{j}}(y,\xi)\right)$ $\forall\, \xi \in [0,1]$.

$\widetilde{\mathbb{T}}_{\mathfrak{f}}\{ \alpha_1(\mathfrak{U}(y)) \oplus \alpha_2(\mathfrak{j}(y))\} = \left(\mathbb{T}\left\{ \alpha_1 \underline{\mathfrak{U}}(y,\xi) + \alpha_2 \underline{\mathfrak{j}}(y,\xi)\right\}, \mathbb{T}\{ \alpha_1 \overline{\mathfrak{U}}(y,\xi) + \alpha_2 \overline{\mathfrak{j}}(y,\xi)\}\right)$

$= \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} (\alpha_1 \underline{\mathfrak{U}}(y,\xi) + \alpha_2 \underline{\mathfrak{j}}(y,\xi))dy$, $\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} (\alpha_1 \overline{\mathfrak{U}}(y,\xi) + \alpha_2 \overline{\mathfrak{j}}(y,\xi))dy$

$= \left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} \alpha_1 \underline{\mathfrak{U}}(y,\xi)dy + \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} \alpha_2 \underline{\mathfrak{j}}(y,\xi)dy\right)$,

$\left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} (\alpha_1 \overline{\mathfrak{U}}(y,\xi)dy + \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} \alpha_2 \overline{\mathfrak{j}}(y,\xi)dy\right) =$

$\alpha_1 \left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} \underline{\mathfrak{U}}(y,\xi)dy, \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} \underline{\mathfrak{U}}(y,\xi)dy\right) + \alpha_2$

$\left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} \underline{\mathfrak{j}}(y,\xi)dy, \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} \overline{\mathfrak{j}}(y,\xi)dy)\right)$

$= \alpha_1 \left(\mathbb{T}\{ \underline{\mathfrak{U}}(y,\xi)\}, \mathbb{T}\{ \overline{\mathfrak{U}}(y,\xi)\}\right) + \alpha_2 \left(\mathbb{T}\left\{ \underline{\mathfrak{j}}(y,\xi)\right\}, \mathbb{T}\{\overline{\mathfrak{j}}(y,\xi)\}\right)$

$= \alpha_1 \odot \widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}(y)\} \oplus \alpha_2 \odot \widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{j}(y)\}$. □

Theorem 5. Consider $\mathfrak{U}: \mathbb{R} \to \mathbb{R}_{\mathfrak{f}}$, $\mathfrak{j}: \mathbb{R} \to \mathbb{R}_{\mathfrak{f}}$ as a continuous fuzzy – valued function, let $\widetilde{\mathbb{F}}(\mathfrak{s},\mathfrak{r}), \widetilde{\mathbb{J}}(\mathfrak{s},\mathfrak{r})$ be fuzzy neoteric transformation for $\mathfrak{U}$ *and* $\mathfrak{j}$ respectively. Then the fuzzy neoteric transformation of the convolution of $\mathfrak{U}$ *and* $\mathfrak{j}$,

$$\widetilde{\mathbb{T}}_{\mathfrak{f}}\{(\mathfrak{U} * \mathfrak{j})(y)\} = \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \widetilde{\mathbb{F}}(\mathfrak{s},\mathfrak{r}) \odot \widetilde{\mathbb{J}}(\mathfrak{s},\mathfrak{r})$$

Proof: let $\widetilde{\mathbb{T}}_{\mathfrak{f}}\{(\mathfrak{U} * \mathfrak{j})(y)\} = \widetilde{\mathbb{T}}_{\mathfrak{f}}\{(\mathbb{H})\}$

$\widetilde{\mathbb{T}}_{\mathfrak{f}}\{(\mathbb{H})\} = \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty [\int_0^y \mathfrak{U}(\tau) \odot \mathfrak{j}(y-\tau)d\tau] e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} dy$

$= (\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty [\int_0^y \underline{\mathfrak{U}}(\tau,\xi). \underline{\mathfrak{j}}(y-\tau,\xi)d\tau] e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} dy, \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty [\int_0^y \overline{\mathfrak{U}}(\tau,\xi). \overline{\mathfrak{j}}(y-\tau,\xi)d\tau] e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} y} dy)$

$= \left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty \underline{\mathfrak{U}}(\tau,\xi) e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\tau} d\tau \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}. \underline{\mathbb{J}}(\mathfrak{s},\mathfrak{r},\xi), \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \int_0^\infty \overline{\mathfrak{U}}(\tau,\xi) e^{-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\tau} d\tau \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}. \overline{\mathbb{J}}(\mathfrak{s},\mathfrak{r},\xi) \right.$, $\forall \xi \in [0,1]$

$= \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \widetilde{\mathbb{F}}(\mathfrak{s}, \mathfrak{r}) \odot \widetilde{\mathbb{J}}(\mathfrak{s}, \mathfrak{r})$ □

4 FUZZY NEOTERIC INTEGRAL FOR ODES

It is important to examine the fuzzy neoteric integral transformation for the derivative of a first and second order function, as well as higher-order derivatives, in order to address fuzzy differential equation . In this context, an application related to first-order ,second- order and n-order of differential equations is presented .

4.1 Fuzzy Neoteric Integral Transformation for first-order differential equation

Assume that $\mathfrak{U}(\omega)$ is a fuzzy function whose derivative is $\mathfrak{U}'(\omega)$. Then, the transform $\widetilde{\mathbb{T}}_f\{\mathfrak{U}'(\omega)\}$ can be represented in terms of $\mathbb{T}\{\mathfrak{U}(\omega)\}$ as follows

Theorem 6: Assume that $\mathfrak{U}(\mathfrak{x})$ denoted an antiderivative of $\mathfrak{U}'(\omega)$ over $[0, \infty)$, where $\mathfrak{U}(\omega)$ is an integrable function taking values in the fuzzy set.

1) if $\mathfrak{U}(\omega)$ is $((a) - \text{differentiable}) \rightarrow \widetilde{\mathbb{T}}_f\{\mathfrak{U}'(\omega)\} = \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \widetilde{\mathbb{T}}_f\{\mathfrak{U}(\mathfrak{x})\} \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \mathfrak{U}(0)$
2) if $\mathfrak{U}(\omega)$ is $((b) - \text{differentiable}) \rightarrow \widetilde{\mathbb{T}}_f\{\mathfrak{U}'(\omega)\} = -\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \mathfrak{U}(0) \ominus (-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}) \widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\}$

Prove 1) $\mathfrak{U}(\omega)$ is $((a) - \text{differentiable}) \ \forall \xi \in [0,1]$

$$\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\} \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \mathfrak{U}(0) = \left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \mathbb{T}\{ \underline{\mathfrak{U}}(\omega, \xi)\} - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \underline{\mathfrak{U}}(0, \xi), \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \mathbb{T}\{ \overline{\mathfrak{U}}(\omega, \xi)\} - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \overline{\mathfrak{U}}(0, \xi) \right) \quad \ldots (1)$$

because

$\mathbb{T}\{\underline{\mathfrak{U}'}(\omega, \xi)\} = \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \mathbb{T}\{\underline{\mathfrak{U}}(\omega, \xi)\} - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \underline{\mathfrak{U}}(0, \xi)$ and $\mathbb{T}\{\overline{\mathfrak{U}}'(\omega, \xi) = \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \mathbb{T}\{\overline{\mathfrak{U}}(\omega, \xi)\} - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} \overline{\mathfrak{U}}(0, \xi)$

Considering that $\mathfrak{U}(\omega)$ is $((a)-differentiable)$, and by theorem 2

$\underline{\mathfrak{U}'}(\omega,\xi)=\underline{\mathfrak{U}'}(\omega,\xi)$, $\overline{\mathfrak{U}'}(\omega,\xi)=\overline{\mathfrak{U}}'(\omega,\xi)$

$\mathbb{T}\{\underline{\mathfrak{U}'}(\omega,\xi)\}=\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\}-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi)$, $\mathbb{T}\{,\overline{\mathfrak{U}'}(\omega,\xi)\ =\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\}-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi)$

So equation (1) become

$\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\}\ominus\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}(0)=(\mathbb{T}\{\underline{\mathfrak{U}'}(\omega,\xi)\},\mathbb{T}\{\overline{\mathfrak{U}'}(\omega,\xi)$ … (2)

But $(\mathbb{T}\{\underline{\mathfrak{U}'}(\omega,\xi)\},\mathbb{T}\{\overline{\mathfrak{U}'}(\omega,\xi)\})=\widetilde{\mathbb{T}}_f\{\mathfrak{U}'(\omega)\}$ ... (3)

Hence by (2) and (3) we get $\widetilde{\mathbb{T}}_f\{\mathfrak{U}'(\omega)\}=\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\widetilde{\mathbb{T}}\{\mathfrak{U}(\omega)\}\ominus\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}(0)$ □

Proof 2) if $\mathfrak{U}(\omega)$ is ((b) − differentiable)

$-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}(0)\ominus(-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})})\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\}=(-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi)+\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\varepsilon)\},-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi)+\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\})$ …(4)

Because

$\mathbb{T}\{\underline{\mathfrak{U}'}(\omega,\xi)\}=\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\}-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi)$ and $\mathbb{T}\{\overline{\mathfrak{U}}'(\omega,\xi)\ =\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\}-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi)$ Considering that $\mathfrak{U}(\omega)$ is $((\mathrm{b})-\text{differentiable})$, and by theorem 2

$\underline{\mathfrak{U}'}(\omega,\xi)=\overline{\mathfrak{U}}'(\omega,\xi)$,$\overline{\mathfrak{U}'}(\omega,\xi)=\underline{\mathfrak{U}'}(\omega,\xi)$

$\mathbb{T}\{\overline{\mathfrak{U}'}(\omega,\xi)\}=\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\}-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi)$, $\mathbb{T}\{\underline{\mathfrak{U}'}(\omega,\xi)\}=\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\}-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi)$

So equation (4) become

$-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}(0)\ominus\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\}=(\mathbb{T}\{\underline{\mathfrak{U}'}(\omega,\xi)\},\mathbb{T}\{\overline{\mathfrak{U}'}(\omega,\xi)\})$ ... (5)

But $(\mathbb{T}\{\underline{\mathfrak{U}'}(\omega,\xi)\},\mathbb{T}\{\overline{\mathfrak{U}'}(\omega,\xi)\})=\widetilde{\mathbb{T}}_f\{\mathfrak{U}'(\omega)\}$ … (6)

Hence by (5) and (6) we get $\widetilde{\mathbb{T}}_f\{\mathfrak{U}'(\omega)\}=-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}(0)\ominus(-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})})\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\}$ □

4.2 Fuzzy neoteric integral transformation for second-order differential equation

Theorem 7. Assume that $\mathfrak{U}(\omega)$ denoted an antiderivative of $\mathfrak{U}''(\omega)$ over $[0,\infty)$, where $\mathfrak{U}(\omega),\mathfrak{U}'(\omega)$ is an integrable function taking values in the fuzzy set.

1) If $\mathfrak{U}(\omega)$ *and* $\mathfrak{U}'(\omega)$ are ((a) − differentiable) →

$$\widetilde{\mathbb{T}}_f\{\mathfrak{U}''(\omega)\} = [\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}\{\mathfrak{U}(\omega)\} \ominus \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0)] \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0)$$

2) If $\mathfrak{U}(\omega)$ is ((a) − differentiable)and $\mathfrak{U}'(\omega)$ is ((b) − differentiable) →

$$\widetilde{\mathbb{T}}_f\{\mathfrak{U}''(\omega)\} = \left(-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0)\right) \ominus [-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\} \ominus \left(-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0)\right)]$$

3) If $\mathfrak{U}(\mathfrak{x})$ is ((b) − differentiable) and $\mathfrak{U}'(\mathfrak{x})$ is ((a) − differentiable) →

$$\widetilde{\mathbb{T}}_f\{\mathfrak{U}''(\omega)\} = \left[-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0) \ominus \left(-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\}\right)\right] \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0).$$

4) If $\mathfrak{U}(\omega)$ and $\mathfrak{U}'(\omega)$are ((b) − differentiable)→

$$\widetilde{\mathbb{T}}_f\{\mathfrak{U}''(\omega)\} = \left(-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0)\right) \ominus \left[\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0) \ominus \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\}\right].$$

Proof :1) let us consider the functions $\mathfrak{U}(\omega)$ and $\mathfrak{U}'(\omega)$ we assume that both functions are ((a) − differentiable)

To proceed , we evaluate the following expression

$[\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\} \ominus \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0)] \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0) = (\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}'}(0,\xi), (\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}}'(0,\xi))$

Since $\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}'(0,\xi)$

$$\mathbb{T}\{\overline{\mathfrak{U}}''(\omega,\xi) = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}}'(0,\xi)$$

And since $\mathfrak{U}(\omega)$, $\mathfrak{U}'(\omega)$ are assumed to be $((a) - \text{differentiable}\ ) \rightarrow \underline{\mathfrak{U}}''(\omega,\xi) = \underline{\mathfrak{U}''}(\omega,\xi)$ and

$\overline{\mathfrak{U}}''(\omega,\xi) = \overline{\mathfrak{U}''}(\omega,\xi)$ (theorem 5)

Moreover, by (theorem 2) the differentiability of $\mathfrak{U}(\omega)$ ensures that $\underline{\mathfrak{U}'}(0,\xi) = \underline{\mathfrak{U}}'(0,\xi)$, $\overline{\mathfrak{U}'}(0,\xi) = \overline{\mathfrak{U}}'(0,\xi)$.

Thus we obtain

$\mathbb{T}\{\underline{\mathfrak{U}''}(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}'}(0,\xi)$,

$\mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}'}(0,\xi)$

Consequently

$[\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\} \ominus \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0)] \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0) = (\mathbb{T}\{\underline{\mathfrak{U}''}(\omega,\varepsilon)\}, \mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\})$ …(7)

But

$\widetilde{\mathbb{T}}_f\{\mathfrak{U}''(\omega)\} = (\mathbb{T}\{\underline{\mathfrak{U}''}(\omega,\xi)\}, \mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\})$ …(8)

Therefore by (7) and (8) we deduce

$\widetilde{\mathbb{T}}_f\{\mathfrak{U}''(\omega)\} = [\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\} \ominus \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0)] \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0)$

Proof: 2) let us consider the functions $\mathfrak{U}(\omega)$ and $\mathfrak{U}'(\omega)$ we assume that, $\mathfrak{U}(\omega)$ is $((a) - \text{differentiable})$ and $\mathfrak{U}'(\omega)$ is $((b) - \text{differentiable}\ )$ To proceed , we evaluate the following expression

$(-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0)) \ominus [-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}(\omega)\} \ominus (-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0))] =$ $(-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}'}(0,\xi) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\overline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\varepsilon), -\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}'}(0,\xi) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\underline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi))$

Since $\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}'(0,\xi)$

$$\mathbb{T}\{\overline{\mathfrak{U}}''(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}}'(0,\xi)$$

And since , $\mathfrak{U}(\omega)$ is $((a)-\text{differentiable})$ and $\mathfrak{U}'(\omega)$ is $((b)-\text{differentiable})$

$\rightarrow \overline{\mathfrak{U}''}(\omega,\xi) = \underline{\mathfrak{U}}''(\omega,\xi)$, $\underline{\mathfrak{U}}''(\omega,\xi) = \overline{\mathfrak{U}}''(\omega,\xi)$ (theorem 5).

Moreover , (by theorem 2), the differentiability of $\mathfrak{U}(\mathfrak{x})$ ensures that $\underline{\mathfrak{U}'}(0,\xi) = \underline{\mathfrak{U}'}(0,\xi)$, $\overline{\mathfrak{U}'}(0,\xi) = \overline{\mathfrak{U}}'(0,\xi)$.

Thus we obtain

$$\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\} = (-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}'}(0,\xi) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi)$$

$$\mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\} = (-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}'}(0,\xi) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi)$$

Consequently

$(-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0)) \ominus [-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}(\omega)\} \ominus (-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0))] =$ $(\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\}, \mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\})$ …(9)

But

$\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}''(\omega)\} = (\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\}, \mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\})$ …(10)

Therefore by (9)and (10) we deduce

$$\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}''(\omega)\} = (-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0)) \ominus [-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}(\omega)\} \ominus (-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0))] .$$

Proof: 3) let us consider the functions $\mathfrak{U}(\omega)$ and $\mathfrak{U}'(\omega)$ we assume that $\mathfrak{U}(\mathfrak{x}\omega)$ is $((b)-\text{differentiable})$ and $\mathfrak{U}'(\omega)$ is $((a)-\text{differentiable})$ To proceed , we evaluate the following expression

$[-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0) \ominus (-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\}] \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0) = (-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi) +$

$\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\} -$

$\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}'(0,\xi)$, $(-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}'}(0,\xi))$

Since , $\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}'(0,\xi)$

$\mathbb{T}\{\overline{\mathfrak{U}}''(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\varepsilon)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\varepsilon) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}}'(0,\varepsilon)$

And since , $\mathfrak{U}(\omega)$ is $(b)-$differentiable and $\mathfrak{U}'(\omega)$ is $((a)-$differentiable)

$\rightarrow \overline{\mathfrak{U}''}(\omega,\xi) = \underline{\mathfrak{U}}''(\omega,\xi)$, $\underline{\mathfrak{U}}''(\omega,\xi) = \overline{\mathfrak{U}}''(\omega,\xi)$ (theorem 5).

Moreover , (by theorem 2), the differentiability of $\mathfrak{U}(\omega)$ ensures that $\overline{\mathfrak{U}}'(0,\xi) = \underline{\mathfrak{U}}'(0,\xi)$, $\underline{\mathfrak{U}}'(0,\xi) = \overline{\mathfrak{U}'}(0,\xi)$.

Thus we obtain

$\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}'(0,\xi)$,

$\mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}'}(0,\xi)$

Consequently

$[-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0) \ominus (-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\}] \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0) = (\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\}, \mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\})$ …(11)

But

$\widetilde{\mathbb{T}}_f\{\mathfrak{U}''(\omega)\} = (\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\}, \mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\})$ …(12)

Therefore by (11)and (12) we deduce

$\widetilde{\mathbb{T}}_f\{\mathfrak{U}''(\omega)\} = [-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0) \ominus (-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\}] \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0)$.

Proof :4) let us consider the functions $\mathfrak{U}(\omega)$ and $\mathfrak{U}'(\omega)$ we assume that both functions are $((b)-$differentiable)

To proceed , we evaluate the following expression

$-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0) \ominus [\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0) \ominus \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathfrak{U}(\omega)\}] = (-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}'}(0,\xi) - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi) +$

$\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}\{\underline{\mathfrak{U}}(\omega,\xi)$, $-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}'(0,\xi) - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\})$

Since $\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}'(0,\xi)$

$$\mathbb{T}\{\overline{\mathfrak{U}}''(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}}'(0,\xi)$$

And since $\mathfrak{U}(\omega)$, $\mathfrak{U}'(\omega)$ are assumed to be $((\mathrm{b}) - \text{differentiable}) \rightarrow \underline{\mathfrak{U}}''(\omega,\xi) = \underline{\mathfrak{U}''}(\omega,\xi)$ and

$\overline{\mathfrak{U}}''(\omega,\xi) = \overline{\mathfrak{U}''}(\omega,\xi)$ (by theorem 5)

Moreover, by (theorem 2) the differentiability of $\mathfrak{U}(\omega)$ ensures that $\overline{\mathfrak{U}'}(0,\xi) = \underline{\mathfrak{U}}'(0,\xi)$, $\underline{\mathfrak{U}'}(0,\xi) = \overline{\mathfrak{U}}'(0,\xi)$.

Thus we obtain

$$\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\underline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\overline{\mathfrak{U}'}(0,\xi),$$

$$\mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\} = \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{\mathfrak{U}}(\omega,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\overline{\mathfrak{U}}(0,\xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\underline{\mathfrak{U}}'(0,\xi)$$

Consequently

$$-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0) \ominus [\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0) \ominus \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}(\omega)\}] = (\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\}, \mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\}) \quad \ldots(13)$$

But

$$\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}''(\omega)\} = (\mathbb{T}\{\underline{\mathfrak{U}}''(\omega,\xi)\}, \mathbb{T}\{\overline{\mathfrak{U}''}(\omega,\xi)\}) \quad \ldots(14)$$

Therefore by (13) and (14) we deduce

$$\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}''(\omega)\} = (-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\mathfrak{U}'(0)) \ominus [\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}(\omega)\} \ominus \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathfrak{U}(0)] \quad \square$$

4.3 Fuzzy neoteric integral transformation for n^{th}-order differential equation

This part explores a universal equation for the fuzzy derivative of any order $n \in Z^+$

Theorem 8:[12] Assume $\mathfrak{U}: (c,d) \rightarrow \mathbb{R}_{\mathfrak{f}}$ and let $y \in (c,d)$. We assert that $\mathbb{f}$ **is strongly generalized differentiab1e** at y_0 if $\exists\, \mathfrak{U}^{\delta}(y) \in \mathbb{R}_{\mathfrak{f}}$, $\forall\, \delta = 1,2,\ldots,n$ s.t. one of the following holds

(a) $\forall\, \text{ꝋ} > 0$, where (ꝋ is small constant), the limits are exist:

$$\lim_{ʉ\to 0}\frac{\mathfrak{U}^{(\delta-1)}(y+ʉ)\ominus\mathfrak{U}^{(\delta-1)}(y)}{ʉ}=\lim_{ʉ\to 0}\frac{\mathfrak{U}^{(\delta-1)}(y)\ominus\mathfrak{U}^{(\delta-1)}(y-ʉ)}{ʉ}=\mathfrak{U}^{(\delta)}(y)$$

(b) $\forall ʉ > 0$, where ($ʉ$ is small constant), the limits are exist:

$$\lim_{ʉ\to 0}\frac{\mathfrak{U}^{(\delta-1)}(y)\ominus\mathfrak{U}^{(\delta-1)}(y+ʉ)}{-ʉ}=\lim_{ʉ\to 0}\frac{\mathfrak{U}^{(\delta-1)}(y-ʉ)\ominus\mathfrak{U}^{(\delta-1)}(y)}{-ʉ}=\mathfrak{U}^{(\delta)}(y)$$

(c) $\forall ʉ > 0$, where (u is small constant), the limits are exist:

$$\lim_{ʉ\to 0}\frac{\mathfrak{U}^{(\delta-1)}(y+ʉ)\ominus\mathfrak{U}^{(\delta-1)}(y)}{ʉ}=\lim_{ʉ\to 0}\frac{\mathfrak{U}^{(\delta-1)}(y-ʉ)\ominus\mathfrak{U}^{(\delta-1)}(y)}{-ʉ}=\mathfrak{U}^{(\delta)}(y)$$

(d) $\forall ʉ > 0$, where (u is small constant), the limits are exist:

$$\lim_{ʉ\to 0}\frac{\mathfrak{U}^{(\delta-1)}(y)\ominus\mathfrak{U}^{(\delta-1)}(y+ʉ)}{-ʉ}=\lim_{ʉ\to 0}\frac{\mathfrak{U}^{(\delta-1)}(y)\ominus\mathfrak{U}^{(\delta-1)}(y-ʉ)}{ʉ}=\mathfrak{U}^{(\delta)}(y)$$

Theorem 9.[12] Assume $\mathfrak{U}(\omega)$, $\mathfrak{U}'(\omega)$, $\mathfrak{U}''(\omega), \dots, \mathfrak{U}^{(n-1)}(\omega)$ be differentiable fuzzy-valued functions. Furthermore, the $\xi -$cut representation of the fuzzy-valued function $\mathfrak{U}(\omega)$is denoted by

$\mathfrak{U}(\omega) = (\underline{\mathfrak{U}}(\omega,\xi)\, , \overline{\mathfrak{U}}(\omega,\xi)\,)\ \ \forall\, \xi \in [0,1]$. Then

$$\mathfrak{U}^{(n)}(\omega)=\begin{bmatrix}\left(\underline{\mathfrak{U}}^{(n)}(\omega,\xi),\overline{\mathfrak{U}}^{(n)}(\omega,\xi)\right) if\ number\ of\ \left((\text{b})-\text{differentiable}\ \right) is\ even\\ \left(\overline{\mathfrak{U}}^{(n)}(\omega,\xi),\underline{\mathfrak{U}}^{(n)}(\omega,\xi)\right)\ if\ number\ of\ ((\text{b})-\text{differentiable})\ is\ odd.\end{bmatrix}$$

Theorem 10. Let $\mathfrak{U}(\omega)$, $\mathfrak{U}'(\omega)$, $\mathfrak{U}''(\omega), \dots, \mathfrak{U}^{(n-1)}(\omega)$ be continuous functions $\forall\, \omega \geq 0$. Assume that $\mathfrak{U}$ is strongly generalized differentiable of order n, so that the elements $\mathfrak{U}^{(\delta)}(\omega_0) \in \mathcal{R}_{\mathfrak{f}}$ exist $\forall\, \delta = 0, \dots n$. Then , the fuzzy Neoteric Integral Transformation of $\mathfrak{U}^{(n)}(\omega)$ given as :

$$\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}^{(n)}(\omega)\} = (\prod_{\mathrm{e}=1}^{n} \mathfrak{S}(\mathrm{e})) \odot \widetilde{\mathbb{T}}_{\mathfrak{f}}[\mathfrak{U}(\omega) \ominus (\prod_{\mathrm{e}=1}^{n} \mathfrak{S}(\mathrm{e})) \odot \mathfrak{U}(0) \ominus \prod_{\mathrm{e}=2}^{n} \mathfrak{S}(\mathrm{e})\mathfrak{U}'(0)$$
$$\ominus \mathfrak{S}(n)\mathfrak{U}^{(n-1)}(0)$$

Such that

$$\mathfrak{S}(\mathrm{e}) = \begin{bmatrix} \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\ if\ \mathfrak{U}^{(\mathrm{e}-1)}\ is\ (a) -\ differentiable \\ \ominus \left(-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\right) if\ \mathfrak{U}^{(\mathrm{e}-1)}\ is\ (b) -\ differentiable \end{bmatrix}$$

Proof : By applying the principle mathematics of induction **,** the subsequent steps are derived:

Step 1 : If n=1 according to theorem 6, the formula is correct

Step 2 :let the formula be true when n=k

$$\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}^{(k)}(\omega)\} = (\prod_{\mathrm{e}=1}^{k} \mathfrak{S}(\mathrm{e})) \odot \widetilde{\mathbb{T}}_{\mathfrak{f}}[\mathfrak{U}(\omega) \ominus (\prod_{\mathrm{e}=1}^{k} \mathfrak{S}(\mathrm{e})) \odot \mathfrak{U}(0) \ominus \prod_{\mathrm{e}=2}^{k} \mathfrak{S}(\mathrm{e})\mathfrak{U}'(0) \ldots$$
$$\ominus \mathfrak{S}(k)\mathfrak{U}^{(k-1)}(0)$$

Since $\widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}^{(k+1)}(\omega)\} = \widetilde{\mathbb{T}}_{\mathfrak{f}}\{(\mathfrak{U}^{(k)})'(\omega)\} = \mathfrak{S}(\mathrm{e}) \odot \widetilde{\mathbb{T}}_{\mathfrak{f}}\{\mathfrak{U}^{(k)}(\omega)\} \ominus \mathfrak{S}(\mathrm{e})\mathfrak{U}^{(k)}(0)$

$$= \mathfrak{S}(\mathrm{e}) \odot [(\prod_{\mathrm{e}=1}^{k} \mathfrak{S}(\mathrm{e})) \odot \widetilde{\mathbb{T}}_{\mathfrak{f}}[\mathfrak{U}(\omega) \ominus (\prod_{\mathrm{e}=1}^{k} \mathfrak{S}(\mathrm{e})) \odot \mathfrak{U}(0) \ominus \prod_{\mathrm{e}=2}^{k} \mathfrak{S}(\mathrm{e})\mathfrak{U}'(0) \ominus \ldots$$
$$\ominus \mathfrak{S}(k)\mathfrak{U}^{(k-1)}(0) \ \ominus \mathfrak{S}(\mathrm{e})\mathfrak{U}^{(k)}(0)$$
$$= (\prod_{\mathrm{e}=1}^{k+1} \mathfrak{S}(\mathrm{e})) \odot \widetilde{\mathbb{T}}_{\mathfrak{f}}[\mathfrak{U}(\omega) \ominus (\prod_{\mathrm{e}=1}^{k+1} \mathfrak{S}(\mathrm{e})) \odot \mathfrak{U}(0) \ominus \prod_{\mathrm{e}=2}^{k+1} \mathfrak{S}(\mathrm{e})\mathfrak{U}'(0) \ominus \ldots$$
$$\ominus \mathfrak{S}(\mathrm{e})\mathfrak{U}^{(k)}(0)$$

Which matches the required formula for $n = k + 1$ □

5 APPLICATION

To demonstrate the efficiency of FNT, it can be used in some applications.

Example (1) : Consider the growth model for the financial audit as

$Ʉ'(y) = kɄ(y),\ y > 0 \quad \tilde{Ʉ}_0 = (\xi - 1, 1 - \xi)$,

$Ʉ(y)$ represents the capital at time y, while k represents the annual interest added to the amount each year (constant). Furthermore, the fuzzy initial condition indicates uncertainty in the opening values of the funds. If assumption k=3 and employ the FNT

$\tilde{\mathbb{T}}_f\{Ʉ'(y)\} = 3\tilde{\mathbb{T}}_f\{Ʉ(y)\}$ from Theorem 6

Case (1) if $Ʉ(y)$ is $((a) - \text{differentiable})$,

$$\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\tilde{\mathbb{T}}_f\{Ʉ(y)\} \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}Ʉ(0,\xi) = 3\tilde{\mathbb{T}}_f\{Ʉ(y)\}$$

Then by using upper and lower mapping and substituting the initial conditions we get

$$\left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} - 3\right)\mathbb{T}\{\,Ʉ(y,\xi)\} = \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\ \{\underline{Ʉ}(0,\xi)\}\ ,\ \left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} - 3\right)\mathbb{T}\left\{\,\overline{Ʉ}(y,\xi)\right\} = \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\ \left\{\overline{Ʉ}(0,\xi)\right\}$$

$$\left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} - 3\right)\mathbb{T}\left\{\,\underline{Ʉ}(y,\xi)\right\} = \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(\xi - 1)\ ,\ \left(\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})} - 3\right)\mathbb{T}\left\{\,\overline{Ʉ}(y,\xi)\right\} = \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(1 - \xi)$$

Simplification

$\underline{Ʉ}(y,\xi) = (\xi - 1)\, e^{3y}\ ,\ \ \overline{Ʉ}(y,\xi) = (1 - \xi)e^{3y}$ □

Case (2) If $Ʉ(y)$ is $((b) - \text{differentiable})$,

$$-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}Ʉ(0,\xi) \ominus \left(-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\right)\tilde{\mathbb{T}}_f\{Ʉ(y)\} = 3\tilde{\mathbb{T}}_f\{Ʉ(y)\}$$

Then by using upper and lower mapping and substituting the initial conditions we get

$$3\mathbb{T}\left\{\,\overline{Ʉ}(y,\xi)\right\} = -\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(\xi - 1) + \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\ \mathbb{T}\left\{\,\underline{Ʉ}(y,\xi)\right\}\quad ,\quad 3\mathbb{T}\left\{\,\underline{Ʉ}(y,\xi)\right\} = -\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(1 - \xi) + \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}\ \mathbb{T}\left\{\,\overline{Ʉ}(y,\xi)\right\}$$

Simplification

$\underline{Ʉ}(y,\xi) = (1 - \xi)\sinh(3\,y) + (\xi - 1)\cosh(3y)$,

$\overline{Ʉ}(y,\xi) = (\xi - 1)\sinh(3\,y) + (1 - \xi)\cosh(3)y$

The fuzzy initial condition expands the range of financial solutions to the differential equation and reveals the possible probabilities for capital within a time period.

Example (2) Consider the differential equation of the deflection curve of abeam is

$Ʉ''(y) = \eta$ with fuzzy initial condition $\mathbb{U}(0,\varepsilon) = \mathbb{U}'(0,\varepsilon) = (\varepsilon - 1, 1 - \varepsilon)$ and η is constant.

Using FNT and Theorem 7

$\widetilde{\mathbb{T}}_f\{Ʉ''(y)\} = \widetilde{\mathbb{T}}_f\{\eta\}$ the following cases get:

Case (1): If $Ʉ(y)$ and $Ʉ'(y)$ are $((a) - \text{differentiable})$

$\widetilde{\mathbb{T}}_f\{Ʉ''(y)\} = \widetilde{\mathbb{T}}_f\{\eta\} \rightarrow \left[\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{\mathbb{U}(\mathbb{t})\} \ominus \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}Ʉ(0)\right] \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}Ʉ'(0) = \widetilde{\mathbb{T}}_f\{\eta\}$.

Then by using upper and lower mapping and substituting the initial conditions we get

$\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{Ʉ}(y,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}(\xi - 1) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(\xi - 1) = \eta,$

$,\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{Ʉ}(y,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}(1 - \xi) - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(1 - \xi) = \eta$

Simplification

$\underline{Ʉ}(y,\varepsilon) = (\xi - 1)[\frac{1}{2}y^2 + y] + \eta$, $\overline{Ʉ}(y,\xi) = (1 - \xi)[\frac{1}{2}y^2 + y] + \eta$.

Case (2): If $Ʉ(y) is$ $((a) - \text{differentiable})$ and $Ʉ'(y)$ is $((b) - \text{differentiable})$

$\widetilde{\mathbb{T}}_f\{Ʉ''(y)\} = \widetilde{\mathbb{T}}_f\{\eta\} \rightarrow -\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}Ʉ'(0) \ominus [-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{Ʉ(y)\} \ominus (-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}Ʉ(0))] = \widetilde{\mathbb{T}}_f\{\eta\}$.

Then by using upper and lower mapping and substituting the initial conditions we get

$-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(1 - \xi) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{Ʉ}(y,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}(1 - \xi) = \eta$,

$-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(\xi - 1) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{Ʉ}(y,\xi)\} - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}(\xi - 1) = \eta.$

Simplification

$\underline{Ʉ}(y,\xi) = (1 - \xi)\left[-\frac{1}{2}y^2 + y\right] + \eta$, $\overline{Ʉ}(y,\xi) = (\xi - 1)\left[-\frac{1}{2}y^2 + y\right] + \eta$.

Case (3): If $Ʉ(y)$ is $((b) - \text{differentiable})$ and $Ʉ'(y)$ is $((a) - \text{differentiable})$

$$\widetilde{\mathbb{T}}_f\{Ʉ''(y)\} = \widetilde{\mathbb{T}}_f\{\eta\} \;\rightarrow\; [-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}Ʉ(0) \ominus [-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{Ʉ(y)\}]] \ominus \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}Ʉ'(0) = \widetilde{\mathbb{T}}_f\{\eta\}$$

Then by using upper and lower mapping and substituting the initial conditions we get

$$-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}(1-\xi) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{Ʉ}(y,\xi)\} - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(\xi-1) = \eta,$$

$$-\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}(\xi-1) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{Ʉ}(y,\xi)\} - \frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(1-\xi) = \eta.$$

Simplification

$$\underline{Ʉ}(y,\xi) = (\xi-1)\left[-\frac{1}{2}y^2 - y\right] + \eta \;\;,\;\; \overline{Ʉ}(y,\xi) = (1-\xi)\left[-\frac{1}{2}y^2 - y\right] + \eta$$

Case (4): If $Ʉ(y)$ and $Ʉ'(y)$are $((b) - \text{differentiable})$

$$\widetilde{\mathbb{T}}_f\{Ʉ''(y)\} = \widetilde{\mathbb{T}}_f\{\eta\} \;\rightarrow -\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}Ʉ'(0) \ominus [\frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}Ʉ(0) \ominus \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\widetilde{\mathbb{T}}_f\{Ʉ(y)\}] = \widetilde{\mathbb{T}}_f\{\eta\}$$

Then by using upper and lower mapping and substituting the initial conditions we get

$$-\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(1-\xi) - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}(\xi-1) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\underline{Ʉ}(y,\xi)\} = \eta\,,$$

$$\frac{\mathfrak{y}(\mathfrak{s})}{\mathbb{v}(\mathfrak{r})}(\varepsilon-1) - \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}(1-\varepsilon) + \frac{\mathfrak{y}^2(\mathfrak{s})}{\mathbb{v}^2(\mathfrak{r})}\mathbb{T}\{\overline{Ʉ}(y,\xi)\} = \eta\,.$$

Simplification

$$\underline{Ʉ}(y,\xi) = (\xi-1)[\frac{1}{2}y^2 - y] + \eta] \,,\; \overline{Ʉ}(y,\xi) = (1-\xi)[\frac{1}{2}y^2 - y] + \eta\,.$$

Solutions for this fuzzy initial conditions model allow the civil engineer to predict potential deviations when load conditions are imprecise, thus raising safety standards in design.

6 CONCLUSION

In this paper, we have extended the neoteric integral transformation to the fuzzy domain. The definition, properties, and potential applications of the fuzzy neoteric integral transform were presented. This approach provides a useful mathematical tool for handling uncertain and imprecise problems, especially in solving fuzzy differential equations. Future work may include further investigation of inversion formulas, convolution theorems, and numerical applications.

REFERENCES

Ahmadi, M. B.; Kiani, N. A.; Mikaeilvand, N. Laplace transform formula on fuzzy nth-order derivative and its application in fuzzy ordinary differential equations. *Soft Computing*, v. 18, p. 2461–2469, 2014.

Albukhuttar, A. N.; Alshamkhawii, J. A.; Kadhim, H. N. Optimizing budget allocation through first-order linear differential equations and innovative transform techniques. *Mathematical Modelling of Engineering Problems*, v. 10, n. 6, p. 2158–2164, 2023.

Bede, B.; Gal, S. G. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets and Systems*, v. 151, p. 581–599, 2005.

Chalco-Cano, Y.; Roman-Flores, H. On new solutions of fuzzy differential equation. *Chaos, Solitons & Fractals*, v. 38, p. 112–119, 2008.

Kadhim, H. N.; Albukhuttar, A. N.; Alibrahimi, M. S. The use of mathematical systems in cost accounting. *AIP Conference Proceedings*, v. 2414, p. 40087, 2023.

Kexue, L.; Jigen, P. Laplace transform and fractional differential equations. *Applied Mathematics Letters*, v. 24, n. 12, 2011.

Layman, J. W. The Hankel transform and some of its properties. *Journal of Integer Sequences*, v. 4, p. 1–11, 2001.

Namias, V. The fractional order Fourier transform and its application to quantum mechanics. *IMA Journal of Applied Mathematics*, v. 25, p. 241–265, 1980.

Saitoh, S. The Weierstrass transform and an isometry in the heat equation. *Applicable Analysis*, v. 16, p. 1–6, 1983.

Salahshour, S.; Allahviranloo, T. Applications of fuzzy Laplace transformation. *Soft Computing*, v. 17, n. 1, p. 145–158, 2013.

Spinelli, R. Numerical inversion of a Laplace transform. *SIAM Journal on Numerical Analysis*, v. 3, p. 636–649, 1966.

Tranter, C. The use of the Mellin transform in finding the stress distribution in an infinite wedge. *Quarterly Journal of Mechanics and Applied Mathematics*, v. 1, p. 125–130, 1948.

Wu, C.; Gong, Z. On Henstock integral of fuzzy-number-valued functions I. *Fuzzy Sets and Systems*, v. 120, p. 523–532, 2001.

Wu, H. C. The fuzzy Riemann integral and its numerical integration. *Fuzzy Sets and Systems*, v. 110, p. 1–5, 2000.

Zainy, Z. R.; Albukhuttar, A. N. Development neoteric general integral transformation via fractional exponent of its kernel. *Preprint*, 2025.

Authors' Contribution

All authors contributed equally to the development of this article.

Data availability

All datasets relevant to this study's findings are fully available within the article.

How to cite this article (APA)

Zainy, Z. R., & Albukhuttar, A. N. (2026). A FUZZY NEOTERIC INTEGRAL TRANSFORMATION. *Veredas Do Direito*, 23(6), e235536. https://doi.org/10.18623/rvd.v23.5536