Artigo original

http://dx.doi.org/10.18623/rvd.v22.n4.3701

SOLVING SOME GEOMETRIC PROBLEMS MORE EFFICIENTLY USING GENERALIZED PTOLEMY'S THEOREM

PROBLEM SOLVING IN ADVANCED EUCLIDIAN GEOMETRY

Article received on: 8/4/2025
Article accepted on: 10/27/2025

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Abstract

If four points A, B, C, D, in this order, lie on a circle O, there is an interesting and useful result, called Ptolemy's Theorem, which links the lengths of the legs of the quadrilateral $ABCD$ with the lengths of its diagonals, by the equality $AB \cdot CD + AD \cdot BC = AC \cdot BD$. This result may be generalized if the points A, B, C, D are replaced by circles that touch the circle O. The result is known as Generalized Ptolemy's Theorem (GPT), also known as Casey's Theorem. This relatively less-known result may be used to prove several geometric results. In this paper, we apply GPT to partially known special problems, demonstrating the efficiency of this proving tool. We also compare it with other ways of proving the results, omitting the GPT. Finally, we also expect that one can pose similar geometric problems, where GPT can efficiently be used as a solving tool.

Keywords: Ptolemy's Theorem. Generalized Ptolemy's Theorem. Problem Solving.

Resumo

Se quatro pontos A, B, C, D, *nesta ordem estiverem situados num círculo* O, *existe um resultado interessante e útil, chamado Teorema de Ptolomeu, que relaciona os comprimentos dos lados do quadrilátero* $ABCD$ *com os comprimentos das suas diagonais, pela igualdade* $AB \cdot CD + AD \cdot BC = AC \cdot BD$. *Este resultado pode ser generalizado se os pontos forem substituídos por círculos que tocam o círculo* O. *O resultado é conhecido como Teorema Generalizado de Ptolomeu (GPT), também conhecido como Teorema de Casey. Este resultado relativamente menos conhecido pode ser usado para provar vários resultados geométricos. Neste artigo, aplicamos o GPT a problemas especiais parcialmente conhecidos, demonstrando a eficiência desta ferramenta de prova. Também o comparamos com outras formas de provar os resultados, omitindo o GPT. Finalmente, também esperamos que se possam colocar problemas geométricos semelhantes, onde o GPT possa ser usado eficientemente como ferramenta de resolução.*

Palavras-chave: *Teorema de Ptolomeu. Teorema de Ptolomeu Generalizado. Resolução de Problemas.*

1 INTRODUCTION

In Euclidean geometry, it is known an interesting relationship between the sides and the diagonals of a convex quadrilateral inscribed in a circle. This is known as Ptolemy's Theorem, and it states:

In every convex quadrilateral inscribed in a circle, the sum of the products of the opposite sides is equal to the product of the diagonals.

This theorem allows generalizations, one of which is known as Casey's Theorem (see Johnson (1960), and Weisstein (1998). One can find detailed information about GPT in Gueron (2002)**.** The focus of this paper is not the proof of GPT for each case; we prove GPT solely in one particular case using Ptolemy's Theorem and then without it. In the following, in the main part of the paper, we address three explicit geometric problems, one of which was presented in Larson (1983), comparing two analytic problem-solving methods: one using GPT and the other without using it. These problems are pure and important geometrical facts. Other applications of GPT may be found in Shirali (1995) and Gonzalez (2011).

2 THEORETICAL FRAMEWORK

2.1 Casey's Theorem revisited

First, from Gueron (2002), the definition of the tangent distance between two oriented circles (cycles) $W_1(O_1, R_1)$ and $W_2(O_2, R_2)$, where O_i, R_i for $i = 1{,}2$ are the centers and the radii of these circles, respectively, is given by the formula

$$d(W_1, W_2) = \sqrt{(O_1O_2)^2 - (R_2 - R_1)^2} \tag{1}$$

or $d(W_1, W_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 - (R_2 - R_1)^2}$ if $O_1(x_1, y_1)$, and $O_2(x_2, y_2)$.

Clearly, this distance can be real positive, imaginary, or zero. We have illustrated these cases with Figures 1, 2, and 3, respectively.

Figure 1

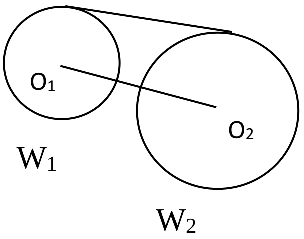


Figure 2

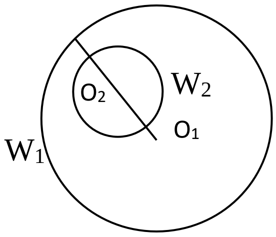
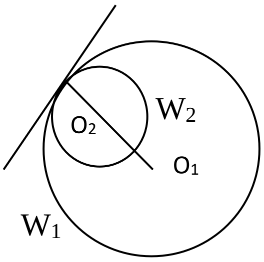


Figure 3



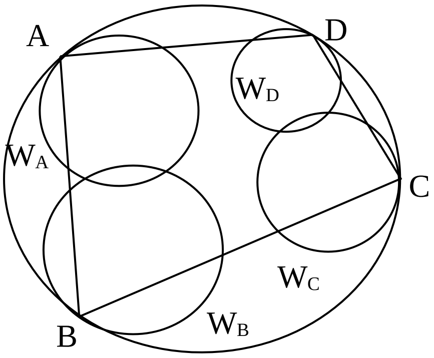
Considering the reciprocal positions between the circles, there are many cases, but our goal is not to prove GPT in all these cases, so we give the proof of GPT just in one special case, first, using Ptolemy's Theorem and then without using it. So, if $d(W_{A,}W_B)$ has the above meaning as the tangent distance between two circles W_A and W_B, then we have the following theorem:

Theorem 1: *If* W_A, W_B, W_C, W_D *are internally or externally tangent circles to the circle* W *at the points* A, B, C, D *respectively, which are the vertices of a convex quadrilateral, then we have:*

$$d(W_A, W_B) \cdot d(W_C, W_D) + d(W_B, W_C) \cdot d(W_D, W_A) = d(W_A, W_C) \cdot d(W_B, W_D) \quad (2)$$

Proof 1: We give the proof solely in the case when the circles W_A, W_B, W_C, W_D are internally tangent to the circle W at the points A, B, C, D respectively, which are the vertices of a convex quadrilateral in this circle (Figure 4). We note that the other cases may be proved similarly. Let R, R_1, R_2, R_3 and R_4 be the radii of circles W, W_A, W_B, W_C, W_D respectively. Using (1), as shown in Gueron (2002), (Lemma 3, point b), in our case, we obtain the following equalities:

Figure 4



$$d(W_A,W_B)=\frac{AB}{R}\sqrt{(R-R_1)(R-R_2)},\ \ d(W_B,W_C)=\frac{BC}{R}\sqrt{(R-R_2)(R-R_3)},$$

$$d(W_C,W_D)=\frac{CD}{R}\sqrt{(R-R_3)(R-R_4)},\ d(W_D,W_A)=\frac{DA}{R}\sqrt{(R-R_4)(R-R_1)},$$

$$d(W_A,W_C)=\tfrac{AC}{R}\sqrt{(R-R_1)(R-R_3)},\ d(W_B,W_D)=\tfrac{BD}{R}\sqrt{(R-R_2)(R-R_4)} \qquad (3)$$

From Ptolemy's Theorem, in the quadrilateral $ABCD$, we have:

$$AB\cdot CD+BC\cdot DA=AC\cdot BD \qquad (4)$$

and from (3) and (4), we obtain equality (2), which is what we wanted to prove. ■

Clearly, this Theorem is a generalization of Ptolemy's Theorem. Indeed, from equality (2), *in the case of circles with zero radii and with centers* A,B,C,D, *we obtain the equality*:

$$AB\cdot CD+BC\cdot DA=AC\cdot BD$$

We see that equality (2) *is also true when some of the circles* W_A,W_B,W_C,W_D *have the zero radii and the others have the radii different from zero*.

Proof 2: We notice that **Theorem 1** is proved using the Ptolemy's Theorem, but it can also be proved without using it. *Indeed*, from equalities (3), we find:

$$d(W_A,W_B)\cdot d(W_C,W_D)+d(W_C,W_B)\cdot d(W_D,W_A)-d(W_A,W_C)\cdot d(W_B,W_D)=$$

$$=(AB\cdot CD+BC\cdot DA-AC\cdot BD)\tfrac{1}{R^2}\sqrt{(R-R_1)(R-R_2)(R-R_3)(R-R_4)} \qquad (5)$$

Now, it suffices to prove that the right side of equality (5) is equal to zero, without wondering for the length of the radii of the circles W_A, W_B, W_C, W_D, (obviously $R_i \neq R$ for $i = 1,2,3,4$). If we show for a special case of the radii R_i, that:

$$d(W_A, W_B).\, d(W_C, W_D) + d(W_C, W_B) \cdot d(W_D, W_A) - d(W_A, W_C) \cdot d(W_B, W_D) = 0$$

we would have:

$$(AB \cdot CD + BC \cdot DA - AC \cdot BD)\frac{1}{R^2}\sqrt{(R - R_1)(R - R_2)(R - R_3)(R - R_4)} = 0,$$

but we always have

$$\frac{1}{R^2}\sqrt{(R - R_1)(R - R_2)(R - R_3)(R - R_4)} \neq 0$$

so, in this special case, we would have the equality $AB \cdot CD + BC \cdot DA - AC \cdot BD = 0$. On the other hand, the expression $AB \cdot CD + BC \cdot DA - AC \cdot BD$ is independent of the radii R_i because the points A, B, C, D are fixed in the circle W. Hence, the equality $AB \cdot CD + BC \cdot DA - AC \cdot BD = 0$ holds true for every R_i; as a result, the right side of equality (5) would always be equal to zero, and hence:

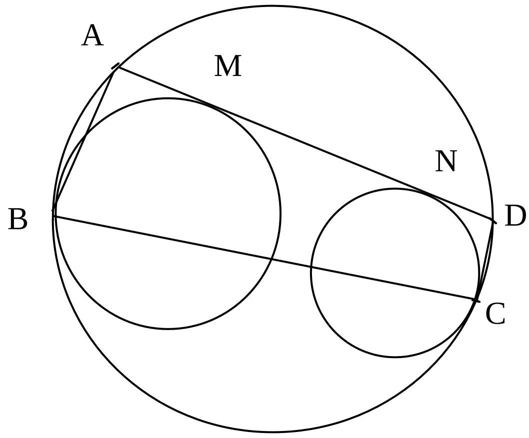
$$d(W_A, W_B) \cdot d(W_C, W_D) + d(W_C, W_B) \cdot d(W_D, W_A) - d(W_A, W_C) \cdot d(W_B, W_D) = 0 \text{ or}$$
$$d(W_A, W_B) \cdot d(W_C, W_D) + d(W_C, W_B) \cdot d(W_D, W_A) = d(W_A, W_C) \cdot d(W_B, W_D)$$

would hold, which would end the proof of GPT.

Now let's consider a special case for what mentioned above, choosing $R_A = R_D = 0$ and the radii $R_B, R_C \neq 0$ (Figure 5) such that the circles W_B, and W_C, are tangent to the cord AD of the circle W. In this case, we have:

$$d(W_B, W_D) = DM, d(W_A, W_B) = AM,\ d(W_C, W_D) = DN,$$
$$d(W_B, W_C) = MN, d(W_D, W_A) = AD,\ d(W_A, W_C) = AN.$$

Figure 5



Hence, it follows that:

$$d(W_A,W_B)\cdot d(W_C,W_D)+d(W_B,W_C)\cdot d(W_D,W_A)-d(W_A,W_C)\cdot d(W_B,W_D)=$$
$$=AM\cdot DN+MN\cdot AD-AN\cdot DM\ =AM\cdot DN+MN\cdot AD-(AM+MN)\cdot DM=$$
$$=AM\cdot(DN-DM)+MN\cdot(AD-DM)=-AM\cdot MN+MN\cdot AM=0$$

So, we have proved GPT without using the Ptolemy's Theorem**.** ■

It also holds, the converse of the GPT, which we give without proof:

Theorem 2: Gonzalez (2011) *Let* W_1, W_2, W_3 *and* W_4 *be four circles, such that*:

$$d(W_1,W_2)\cdot d(W_3,W_4)+d(W_2,W_3)\cdot d(W_4,W_1)=d(W_1,W_3)\cdot d(W_2,W_4)$$

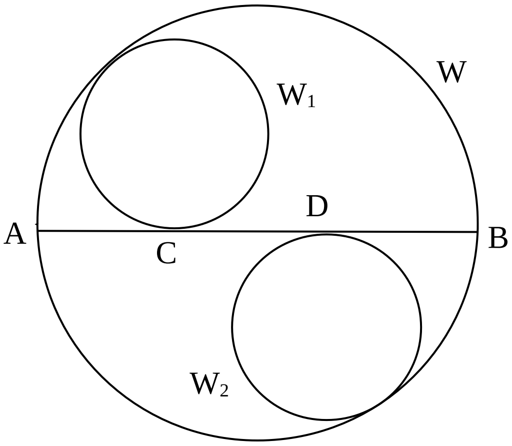
where $d\left(W_i,W_j\right)$, for $i\neq j\ (i,j=1,2,3,4)$ *are the lengths of the common tangents of circles* W_i, W_j *(externally or internally) under the rule that in each three circles there must be three common externally tangent or one externally and the other two internally tangents. Then, there exists a circle that is tangent to the four given circles.* ■

4 RESULTS AND DISCUSSIONS

Now, using **Theorem 1,** we can solve some problems in a simpler, more efficient and quicker way. In order to show this, we have solved the following problems in two ways, one using GPT, and one without using it.

Problem 1: *Let W be a circle with radius R. Let C and D divide its diameter AB into three equal parts (Figure 6). The circles* W_1 *and* W_2 *are tangent to AB at the points C and D, internally tangent to W and are placed in different sides of AB. Find* $d(W_1, W_2)$.

Figure 6



Solution 1: (Using GPT) For the four circles W_1, A, W_2 and B, we can apply **Theorem 1** and we have:

$$d(A,W_1) \cdot d(B,W_2) + d(A,W_2) \cdot d(B,W_1) = d(W_1,W_2) \cdot AB$$

Or

$$AC \cdot BD + AD \cdot BC = d(W_1,W_2) \cdot AB.$$

But $AB = 2R, AC = BD = \frac{1}{3} \cdot AB = \frac{2R}{3},\ AD = BC = \frac{2}{3} \cdot AB = \frac{4R}{3}$. So, we have

$$\left(\frac{2R}{3}\right)^2 + \left(\frac{4R}{3}\right)^2 = d(W_1,W_2) \cdot 2R$$

which gives $d(W_1,W_2) = \frac{10}{9}R$ ■

Solution 2: (Without using GPT)

$d(W_1,W_2) = MN =$? (Figure7).

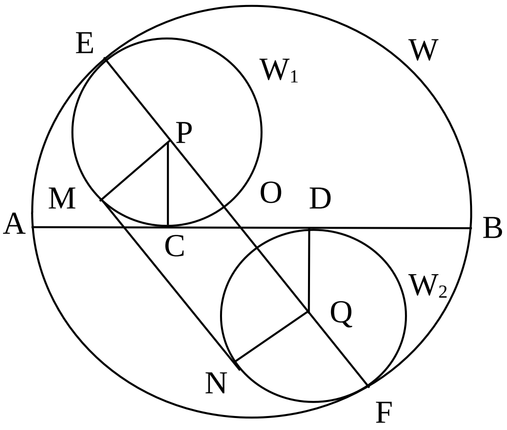
Let r be the radius of the circles W_1 and W_2. The quadrilateral $MNQP$ is a rectangle because $MP = NQ = r$ and $\measuredangle PMN = \measuredangle QNM = 90^0$. We also notice that EF

is the diameter of the circle W and it passes through the centers of the circles W_1 e W_2. So,

$PQ = EF - (EP + FQ) = 2R - 2r = 2(R - r)$.

On the other hand, we have: $PQ = 2OP = 2\sqrt{r^2 + (\frac{R}{3})^2} = \frac{2}{3}\sqrt{9r^2 + R^2}$, from which we find that: $2(R - r) = \frac{2}{3}\sqrt{9r^2 + R^2}$or $r = \frac{4R}{9}$. Finally, we have: $d(W_1, W_2) = MN = PQ = 2R - 2r = 2R - \frac{8R}{9} = \frac{10R}{9}$ ■

Figure 7



Comments on the methods of the problem solution:

1) Even though the second solution is relatively quicker and not very dificult, it is still evident the usefullness and efficiency of the Generalized Ptolemy's Theorem to solve the problem in a quicker and easier way.
2) In the first solution it is obvious the fact that the conditions of the Generalized Ptolemy's Thorem are satified and applying it, we find directly $d(W_1, W_2)$.
3) In the second solution, it is constructed the diameter EF (as a helpful construction), however it must be explained why EF is a diameter and why it goes through the centers P and Q of the circles W_1 and W_2 respectively.
4) In the second solution, PQ must be expressed in two ways in order to find r in terms of R and then to find $d(W_1, W_2)$.

Problem 2: Let W_1, W_2 and W_3 be three congruent circles that are internally tangent to W and externally tangent to each other (Figure 8). Let A be a point on circle

W from where are drawn the tangents to circles W_1, W_2 and W_3. Prove that the length of one of these tangents equals to the sum of the others two.

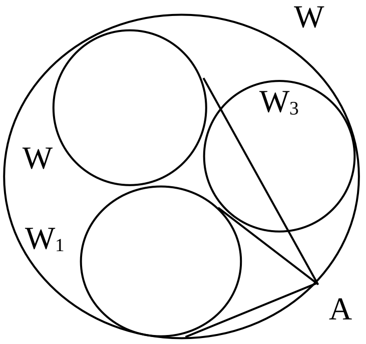
Solution 1: (Using GPT) Since the circles W_1, W_2, W_3 are congruent, we have: $d(W_1, W_2) = d(W_2, W_3) = d(W_1, W_3) = a$. Now using **Theorem 1** for the circles W_1, W_2, W_3 and A, where A is a circle with zero radius, we have:

$$d(W_1, W_2) \cdot d(A, W_3) + d(W_2, W_3) \cdot d(A, W_1) = d(W_1, W_3) \cdot d(A, W_2)$$

Or

$$a \cdot d(A, W_3) + a \cdot d(A, W_1) = a \cdot d(A, W_2)$$

Figure 8

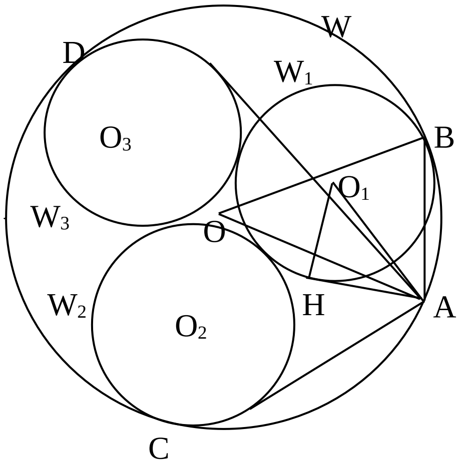


from where it follows that, $d(A, W_3) + d(A, W_1) = d(A, W_2)$: hence, proved. ■

Solution 2: (Without using GPT)

Denote $AB = x_1,\ AC = x_2,\ AD = x_3,\ AO_1 = y_1,\ AO_1 = y_1,\ AO_2 = y_2,\ AO_3 = y_3,\ R$ radius of circle W, r- radius of circles W_1, W_2, W_3 and $d(A, W_1) = d_1 = AH$, $d(A, W_2) = d_2$, $d(A, W_3) = d_3$ (Figure 9).

Figure 9



Let us prove that: $d(A,W_1)+d(A,W_2)=d(A,W_3)$ or $d_1+d_2=d_3$. We see from the figure that: $OA=OB=R$, $OO_1=R-r$. Let us denote $\angle AOB=\theta$; from the cosine theorem in triangles AOB and AOO_1, we have:

$$AB^2=x_1^2=2R^2-2R^2\cdot\cos\theta \text{ and } AO_1^2=y_1^2=R^2+(R-r)^2-2R(R-r)\cdot\cos\theta.$$

Finding $\cos\theta$ from the first equality and by substituting it in the second one we find:

$$y_1^2=R^2+(R-r)^2-2R(R-r)\cdot\frac{2R^2-x_1^2}{2R^2}$$

or $y_1^2=R^2+(R-r)^2-(R-r)\cdot\frac{2R^2-x_1^2}{R}$. In the right triangle AO_1H we have: $AH^2={AO_1}^2-O_1H^2$ or ${d_1}^2=y_1^2-r^2$. So, by substituting y_1^2 from above, we find:

$$d_1^2=R^2+(R-r)^2-(R-r)\cdot\frac{2R^2-x_1^2}{R}-r^2=2R^2-2Rr-(R-r)\cdot\frac{2R^2-x_1^2}{R}$$

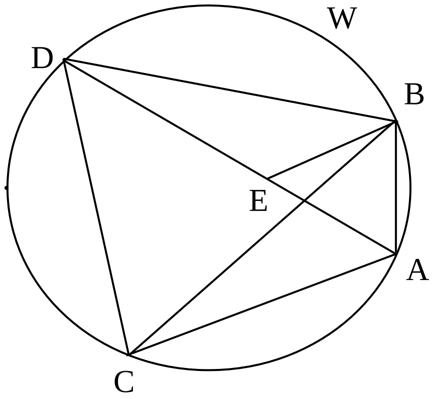
$$=2R(R-r)-(R-r)\cdot\frac{2R^2-x_1^2}{R}=(R-r)[2R-\frac{2R^2-x_1^2}{R}]=x_1^2\cdot\frac{R-r}{R}$$

Finally, we find: $d_1=x_1\cdot\sqrt{\frac{R-r}{R}}$ and similarly: $d_2=x_2\cdot\sqrt{\frac{R-r}{R}}$, $d_3=x_3\cdot\sqrt{\frac{R-r}{R}}$. So, in order to prove $d_1+d_2=d_3$, it suffices to prove that:

$x_1\cdot\sqrt{\frac{R-r}{R}}+x_2\cdot\sqrt{\frac{R-r}{R}}=x_3\cdot\sqrt{\frac{R-r}{R}}$ or $x_1+x_2=x_3$.

The last equality is a well-known problem where x_1, x_2 and x_3 express the distances of the point A in the circumscribed circle of the equilateral triangle BCD from its vertices respectively, (Figure 10). We will prove this as an auxiliary problem:

Figure 10



Let us prove that $x_1 + x_2 = x_3$ or $AB + AC = AD$. Take $AE = AB$, from where it follows that the triangle ABE is also equilateral, since $\measuredangle BAD = 60^0$. So $\measuredangle BED = 120^0$ and $\Delta ABC = \Delta BED$, hence $AC = ED$. So, finally we have: $AB + AC = AD$ or $d(A, W_1) + d(A, W_2) = d(A, W_3)$ ■

Comments on the methods of the problem solution:

1) If the GPT is known, one can immediately notice that its conditions are sastisfied; hence we reach very fastly to the solution.

2) In the second method, in order to find out the relationships between d_1, d_2, it is harder to orientate in figure 9.

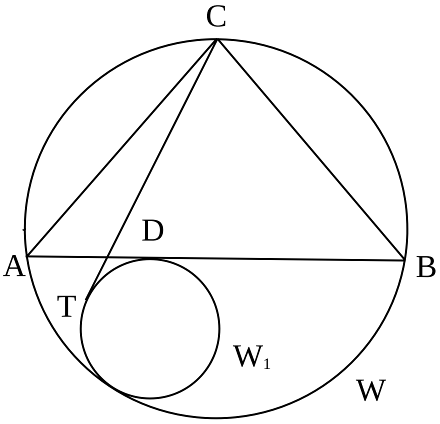
3) In the second method, the transformations are long and complicated to lead to the auxiliary problem.

4) In the second method, in order to reach the conclusion, we also have to solve an auxiliary problem, which complicates the situation and makes the second method longer than the first one.

Problem 3: (Generalization of Problem 8.1.15 taken from Larson (1983)) *The isosceles triangle ABC ($AC = BC$) is inscribed in the circle W (Figure 11). The circle W_1 is tangent to the chord AB and to the arc of the circle W that is determined from AB but* *which does not contain point* C*. Tangent* CT *is drawn from point* C *to the circle* W_1 (*T is the point of tangency*). *Prove that* $\mathrm{CT} = \mathrm{AC} = \mathrm{BC}$.

Solution 1: (Using GPT)

Figure 11



Let us consider four circles A, C, B and W_1 from which the first three have zero radii. Using **Theorem 1** we have:

$$AC \cdot BD + BC \cdot DA = AB \cdot CT \tag{6}$$

where D is the point of tangency of W_1 on AB. From $AC = BC$ and (6) we have $AC \cdot (BD + DA) = AB \cdot CT$ or $AC \cdot AB = AB \cdot CT$ so

$$CT = AC = BC \tag{7}$$

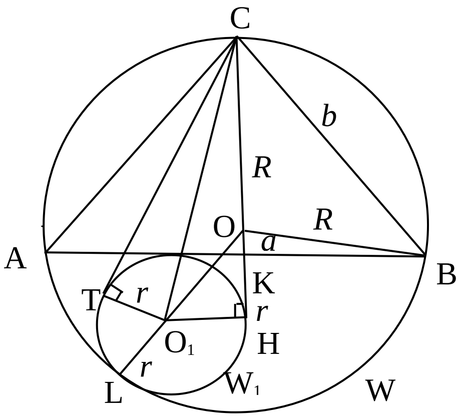
From (7), it follows that T is placed in the arc of the circle with center C and radius CA from A to B. ■

Solution 2: (Without using GPT) (Figure 12)

Denote $OK = a, CA = CB = b$ and radii of circles W and W_1 with R and r respectively. In triangle CTO_1 we have:

$$CT^2 = CO_1{}^2 - r^2 \tag{8}$$

Figure 12



whereas in the triangle CHO_1 we have:

$$CO_1{}^2 = CH^2 + O_1H^2 = (R + a + r)^2 + (O_1O^2 - OH^2) =$$
$$= (R + a + r)^2 + [(R - r)^2 - (a + r)^2].$$

After calculations, we find:

$$CO_1{}^2 = 2R^2 + r^2 + 2Ra \qquad (9)$$

In triangles CKB and OKB we have:

$KB^2 = CB^2 - CK^2$ and $KB^2 = OB^2 - OK^2$

Or

$KB^2 = b^2 - (R + a)^2$ and $KB^2 = R^2 - a^2$

from which we find:

$$a = \frac{b^2}{2R} - R \qquad (10)$$

Finally, from equalities (8), (9) and (10) we take:

$CT^2 = (2R^2 + r^2 + 2Ra) - r^2 = 2R^2 + 2Ra = 2R^2 + 2R(\frac{b^2}{2R} - R) = \quad = b^2 = CA^2 = CB^2$ or $CT = CA = CB$ which ends the solution of the problem. ■

Comments on the two solutions introduced above:

1. The solution of this problem using GPT is quicker, shorter and very easy to understand.
2. The only difficulty that can be encountered in the first solution of the problem (the one where GPT is used), is to understand the opportunity to use this theorem, given the fact that three of four internally tangent circles with the given circle are points (circles with zero radii).
3. In the second solution (without using GPT), some auxiliary constructions must be done, which, in geometry are relatively hard to think of.
4. The number of operations to achieve the final result is large; so, as we move along the solution, the chances to make mistakes with simple operations increases.

5 CONCLUSIONS

From the material presented in this paper, it can be noticed that GPT is a very powerful means for the proof of different geometrical facts, in those cases where it is applicable. Here, we introduced three problems, in order to show the efficiency of this theorem. This is why we decided to give e second solution to each of these really hard problems wich have come up in world class competitions, without using GPT.

Obviously, the second solution is one of the many ways that these problems can be solved (easier or harder), but one clearly notice that in all the problems, the second solution involves auxilary constructions, which are relatively hard ideas to come up with, in order to lead to right direction to finding the final solution. On the other hand, the transformations after these auxilary constructions get even more complicated, complex and longer.

The efficiency of GPT increases, because of the flexbility of its applications in the special cases when one, two or three of the circles tangent to a given circle are with radii zero, meaning they are points. So, it suffices to notice the existence of at least one tangent circle with a given circle, in order to get an indication that the Generalized Ptolemy's Theorem can be applied.

From the above problems , there is one thing that can be seen throughout all the solutions when using GPT. A lot of attention has to be paid when writing the equality which expresses this theorem, and choose according to the given case the appropriate tangent. After this, the rest of the solution is too easy.

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Authors' Contribution

Both authors contributed equally to the development of this article.

Data availability

All datasets relevant to this study's findings are fully available within the article.

How to cite this article (APA)

Hysa, O., & Beqiri, O. (2025). SOLVING SOME GEOMETRIC PROBLEMS MORE EFFICIENTLY USING GENERALIZED PTOLEMY'S THEOREM. Veredas Do Direito, 22(4), e223701. https://doi.org/10.18623/rvd.v22.n4.3701